

PROBLEMS

1. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying

$$f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}.$$

Suppose that f is continuous at 0 . Prove that f is linear, i.e., $f(x) = cx$ for some constant $c \in \mathbb{R}$.

SOLUTIONS

1. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Suppose that f is continuous at 0 . We will show that $f(x) = cx$ for some constant $c \in \mathbb{R}$.

First, we show that $f(0) = 0$. Let $x = 0$ and $y = 0$ in the functional equation. Then $f(0+0) = f(0) + f(0)$, which implies $f(0) = 2f(0)$. Hence $f(0) = 0$.

Next, we show that $f(x) = cx$ for all $x \in \mathbb{R}$. Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then $f(nx) = f(x+x+\dots+x) = n f(x)$. Hence $f(x) = \frac{f(nx)}{n}$. Since f is continuous at 0 , we have $\lim_{n \rightarrow \infty} f(\frac{x}{n}) = f(0) = 0$. Therefore $\lim_{n \rightarrow \infty} \frac{f(nx)}{n} = 0$. This implies $f(x) = 0$ for all $x \in \mathbb{R}$.

Thus, $f(x) = 0$ for all $x \in \mathbb{R}$. This is a linear function with $c = 0$.